CONVERGENCE OF OPEN MAPPINGS OF LOCALLY COMPACT SPACES

BY

SHIGERU ISHII

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY SHIGERU ISHII

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S. S. Cairns
In Charge of Thesis

S. S. Cairns
Head of Department

Recommendation concurred in†

S. S. Cairns
D G Bourgin
P W Ketchum
A. K. Jurgutter
M. K. Fort Jr.

Committee on Final Examination†

† Required for doctor's degree but not for master's

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Let $H(X)$ be a topological group of homeomorphisms of a space X. Then by definition, the mapping $h \rightarrow h^{-1}$ of $H(X)$ onto itself is continuous, i.e. whenever $h = \lim h_\delta$, we have $h^{-1} = \lim h_\delta^{-1}$. This requirement of a topological group suggests the problem of defining a type of convergence in a wider class $C(X,Y)$ of continuous functions on a space X into another Y, the convergence being of such a nature that a convergence property similar to that of the inverse operation of a topological group also holds. As a natural extension of the concept of the inverse of a homeomorphism, the "inverse" associated with a function f in $C(X,Y)$ can be taken to mean the transformation f^{-1} of Y into the set 2^X of all closed subsets of X, where the value $f^{-1}(y)$ of f^{-1} at a point $y \in Y$ is the set of all $x \in X$ such that $y = f(x)$. This definition however entails a new problem of selecting a suitable topology or limit operation in 2^X, if in prescribing a form of convergence for the inverse transformations we are to employ pointwise convergence, continuous convergence, or the like.

One such limit operation is P. Painlevé's familiar limit operation for a sequence of closed sets, generalized to a directed set of closed sets, where $F = \mathrm{Lim}\, F_\sigma$ is defined to mean that $F = \mathrm{Lim\,inf}\, F_\sigma = \mathrm{Lim\,sup}\, F_\sigma$. With this choice, the inverse of an interior (= open) mapping $f: X \rightarrow Y$ is a continuous function on Y into 2^X in the sense that $f^{-1}(y) = \mathrm{Lim}\, f^{-1}(y_\sigma)$ whenever $y = \lim y_\sigma$ and $y_\sigma \in f(X)$ for each σ. This is a generalization of Eilenberg's criterion for interiority of functions on metric spaces. With this result in mind, a type of convergence, called "interior" convergence is defined in this paper as follows: $f_\delta \rightarrow f_0$ interiorly provided 1) whenever $x_0 = \lim_{\Sigma} x_\sigma$, we have $f_0(x_0) = \lim_{\Delta\times\Sigma} f_\delta(x_\sigma)$, i.e. $f_\delta \rightarrow f_0$ continuously, and 2) whenever $y_0 = \lim_{\Delta\times\Sigma} y_{\delta\sigma}$, where $y_{\delta\sigma} \varepsilon f_\delta(X)$ for each $\delta \varepsilon \Delta$, $\operatorname{Lim}_{\Delta\times\Sigma} f_\delta^{-1}(y_{\delta\sigma})$ exists. It turns out that if $f_\delta \rightarrow f_0$ interiorly, where f_δ, f_0 are mappings of a locally compact Hausdorff space into a Hausdorff space, then f_0 is an interior mapping. Note that this result is independent of the interiority of the members f_δ of the directed set and is therefore, in view of Eilenberg's criterion, analogous to the proposition that if $f_\delta \rightarrow f_0$ continuously, f_δ, f_0 being transformations into a regular space, then f_0 is continuous.

Having obtained this result, our attention is restricted to interior convergence within the set $I(X,Y)$ of interior mappings of one space X into another Y. The question which arises now is whether or not there is a topology for $I(X,Y)$ such that convergence in the topological sense is consistent with interior convergence. For most spaces, $I(X,Y)$ is too large a class to admit such a topology, but in the subset $I^*(X,Y)$ of strongly interior mappings of a locally compact space X into a locally connected space Y, there is a topology, here called the "interior" topology, which meets our requirement. When X is locally compact, $I^*(X,Y)$ under the interior topology is completely regular. In the set $L(X,Y)$ of light strongly interior mappings of a locally compact space X into a locally connected space Y, the interior topology coincides with the compact-open topology. In the set $H(X)$ of homeomorphisms of X onto itself, the interior topology coincides with R. Arens' g-topology.

A generalization of G. T. Whyburn's topological analog of the Weierstrass double series theorem is proved and utilized in the proof of the proposition stated above for $L(X,Y)$. This in turn yields a generalization of a theorem of Hurwitz concerning analytic functions of a complex variable.

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INTRODUCTION

Let $C(X,Y)$ be a set of continuous functions on a topological space X into another Y. If K and U are subsets of X and Y respectively, then let $M(K,U)$ denote the set of functions f in $C(X,Y)$ such that $f(K) \subset U$. The compact-open topology for $C(X,Y)$ is then defined by taking as an open subbasis, the totality of sets $M(K,U)$, where K is a compact subset of X and U is an open subset of Y. A modification of the compact-open topology for the case where $X = Y$, the g-topology due to R. Arens [3][1], is obtained by similarly prescribing sets $M(K,U)$ as subbasic open sets, where now K is a closed and U is an open subset of X, and either K or the complement of U in X is compact. Among other results, Arens has established the following proposition relative to convergence properties of the compact-open and g-topologies for $H(X)$, the homeomorphisms of a locally compact Hausdorff space X onto itself (Arens [3, Theorem 5, ii]):

$$h = \lim h_\nu \text{ in the g-topology if and only if}$$
$$h = \lim h_\nu \text{ and } h^{-1} = \lim h_\nu^{-1} \text{ in the compact-open topology.}$$

Arens' theorem suggests the problem of defining a type of convergence in a class $C(X,Y)$ of continuous functions, possibly containing the homeomorphisms of X onto Y, the convergence being of such a nature that a convergence property similar to that of the above theorem also holds. A function f in $C(X,Y)$ may be many-to-one and therefore the associated inverse f^{-1} may not be a function on Y into X in the usual sense of the word "function", but f^{-1} can be interpreted as a transformation on Y into the set 2^X of closed subsets of X.

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[1] Numerals enclosed in brackets refer to the bibliography located at the end of this paper.

Such an interpretation however entails a new problem of defining a suitable topology or limit operation in 2^X, if in prescribing a form of convergence for the inverse transformations, we are to employ pointwise convergence, continuous convergence, or the like. Of course, given any topology t for 2^X such that X is homeomorphically imbedded in $(2^X,t)$ under the natural mapping $x \rightarrow \{x\}$, a type of convergence in $C(X,Y)$ which automatically satisfies the conditions of our original problem is definable--Arens' theorem can be taken as a definition of convergence: $f_\nu \rightarrow f$ if and only if $f = \lim f_\nu$ and $f^{-1} = \lim f_\nu^{-1}$ in the compact-open topology. A closely related method is adopted in this paper but the choice of a topology for 2^X is naturally dictated by considerations other than that of extending Arens' result.

As regards the content and results of this paper, in I, a few notational and terminological conventions are fixed and some known theorems are listed for later reference. II is devoted to an investigation of a topology for 2^X alluded to in the preceding paragraph. In III, the original problem of defining a type of convergence with the desired property is taken up and there a form of convergence, called "interior convergence" is defined and some elementary properties of interior convergence are established. The possibility of introducing a topology for a certain class of continuous functions so that convergence in this topology is equivalent to interior convergence is also considered in III. IV is addressed for the most part to properties relating interior convergence and convergence in the compact-open topology within certain sets of functions into locally connected spaces. This section follows closely one aspect of G. T. Whyburn's investigations of open mappings and eventually leads to a generalization of a classical theorem on analytic functions of a complex variable.

I. PRELIMINARIES

1.1 <u>Definition</u>. A <u>directed system</u> is a non-empty set Δ in which a binary relation $<$ is defined, subject to the following conditions:

(a) If $\delta_1 < \delta_2$ and $\delta_2 < \delta_3$ then $\delta_1 < \delta_3$.

(b) If δ_1 and δ_2 belong to Δ, there exists a δ_3 in Δ such that $\delta_1 < \delta_3$ and $\delta_2 < \delta_3$.

1.2 <u>Definition</u>. A subset Δ' of Δ is <u>cofinal</u> in Δ, provided for each $\delta \in \Delta$, there exists a $\delta' \in \Delta'$ such that $\delta < \delta'$.

If $\Delta = \Delta_1 \cup \Delta_2$ then either Δ_1 or Δ_2 is cofinal in Δ. Hence if $\Delta = \bigcup_{i=1}^{n} \Delta_i$, then Δ_i is cofinal in Δ for some i, $1 \leq i \leq n$.

1.3 <u>Definition</u>. The <u>product</u> $\Delta \times \Sigma$ of two directed systems, Δ and Σ, is the directed system consisting of the set of all ordered pairs (δ, σ) of elements $\delta \in \Delta$ and $\sigma \in \Sigma$, directed by the relation $<$ defined by

$(\delta_1, \sigma_1) < (\delta_2, \sigma_2)$ if and only if $\delta_1 < \delta_2$ and $\sigma_1 < \sigma_2$.

1.4 <u>Definition</u>. If X is a topological space and f is any function on a directed system Δ into X, a point $x \in X$ is a <u>limit point</u> of f, $x = \lim_{\Delta} f(\delta)$, provided for every neighborhood U of x, there exists a $\delta_0 \in \Delta$ such that $f(\delta) \in U$ whenever $\delta_0 < \delta$. If such a point exists, f is said to <u>converge</u> to x, or simply, to be <u>convergent</u>.

The value of a function f on a directed system Δ at a point δ in Δ is usually indicated by f_δ, $f_\delta = f(\delta)$.

1.5 <u>Definition</u>. A point x of a topological space X is a <u>cluster point</u> of a function f on a directed system Δ into X, provided for each neighborhood U of x, the set $f^{-1}(U)$ is cofinal in Δ. A function f on a directed system Δ into a topological space X is an <u>ultrafunction</u> in X if for every open subset U of X, either $f^{-1}(U)$ or $f^{-1}(X - U)$ is not cofinal in Δ.

1.6 <u>Theorem</u>. Each of the following conditions is equivalent to compactness of a space X:

(a) Every function on a directed system into X has a cluster point.

(b) Every ultrafunction in X is convergent.

For a proof see J. L. Kelley [9], or lecture notes of M. M. Day.

Let T(X,Y) be a set of transformations on a topological space X into another Y.

1.7 <u>Definition</u>. A function f on a directed system Δ into T(X,Y) is said to <u>converge continuously</u> to $f_0 \in T(X,Y)$, $f_\delta \to f_0$ continuously, provided for every directed system Σ and function x on Σ into X, if $x_0 = \lim_{\Sigma} x_\sigma$, then $f_0(x_0) = \lim_{\Delta\times\Sigma} f_\delta(x_\sigma)$; or equivalently, for every $x_0 \in X$ and neighborhood V of $f_0(x_0)$, there exists a neighborhood U of x_0 and a $\delta_0 \in \Delta$ such that $f_\delta(U) \subset V$ whenever $\delta_0 < \delta$.

1.8 <u>Theorem</u>. Let Y be a regular space. If f: $\Delta \to T(X,Y)$ converges continuously to $f_0 \in T(X,Y)$, then f_0 is a continuous function.

See O. Frink [8] for a proof. Note that continuity of the functions $f(\delta) = f_\delta$: $X \to Y$ is not presupposed in this theorem. Compare with Theorem 3.6 of this paper.

1.9 <u>Theorem</u>. Let $C(X,Y)$ be a set of continuous functions on a space X into another Y, and let $f\colon \Delta \rightarrow C(X,Y)$ be a function on a directed system into $C(X,Y)$.

(a) If $f_\delta = f_0$ for each $\delta \varepsilon \Delta$, then $f_\delta \rightarrow f_0$ continuously.

(b) If $f_\delta \rightarrow f_0$ continuously and Δ' is cofinal in Δ, then $f_\delta \underset{\Delta'}{\rightarrow} f_0$ continuously.

1.10 <u>Theorem</u>. Let f be a function on a directed system Δ into a set $C(X,Y)$ of continuous functions on a space X into Y.

(a) If $f_\delta \rightarrow f_0$ continuously, then $f_0 = \lim\limits_{\Delta} f_\delta$ in the compact-open topology.

(b) If X is a locally compact Hausdorff space, then $f_\delta \rightarrow f_0$ continuously if and only if $f_0 = \lim\limits_{\Delta} f_\delta$ in the compact-open topology.

See R. Arens [2] for a proof of 1.10.

II. THE SET OF CLOSED SUBSETS OF A SPACE

The need for a topology or some limit operation in the set of all closed subsets of a space X for our purpose has been indicated in the Introduction. From among the various topologies for 2^X extant in the literature, we shall select one due to G. Choquet [6]. Choquet's definition and results are explicitly stated for the set of non-empty closed subsets of a space. But among the functions $f: X \rightarrow Y$ we are to consider, the associated inverse f^{-1} may take a point $y \varepsilon Y$ into the empty set $\emptyset$, $\emptyset = f^{-1}(y)$. To accommodate the empty set, we shall take Choquet's definition and simply extend it to the set of all closed subsets of a space. The inclusion of $\emptyset$ into our schema gives rise to some changes in results. For this reason, it does not seem amiss to accord a separate treatment here to this topology for 2^X.

A study of topologies for 2^X is not without its own intrinsic interest. From this point of view, aside from our particular desire to consider the inverse f^{-1} of a continuous function $f: X \rightarrow Y$ as defined on the entire space Y, as far as Choquet's topology is concerned, the inclusion of $\emptyset$ as an object of interest will be seen to possess the especial virtue of serving to simplify the proofs of certain theorems by allowing the use of the familiar limit operation of P. Painlevé in 2^X (Kuratowski [10, p. 241], O. Frink [8]). The intimate relationship subsisting between this limit operation and the particular topology for 2^X we have selected is not accidental. In fact this topology resulted from an attempt to construct a topology for 2^X such that limits in 2^X in the topological sense is equivalent to the Painlevé limit operation which was given beforehand. We now turn to this limit operation. It is defined within the set of all subsets of a space.

Let $F: \Delta \to \mathcal{A}(X)$ be a function on a directed system Δ into the set $\mathcal{A}(X)$ of all subsets of a topological space X.

2.1 Definition. The limit superior of F, written $\operatorname{Lim}_{\Delta}\sup F_\delta$, is the set of all $x \in X$ such that for every neighborhood U of x and every $\delta \in \Delta$ there exists a $\delta' \in \Delta$ such that $\delta < \delta'$ and $U \cap F_{\delta'} \neq \emptyset$. That is to say $x \in \operatorname{Lim}_{\Delta}\sup F_\delta$ if and only if for every neighborhood U of x, the set

$$A(U) = \{\delta \mid \delta \in \Delta \ \& \ U \cap F_\delta \neq \emptyset\}$$

is cofinal in Δ.

2.2 Definition. The limit inferior of F, written $\operatorname{Lim}_{\Delta}\inf F_\delta$, is the set of all $x \in X$ such that for every neighborhood U of x, there exists a $\delta \in \Delta$, such that if $\delta < \delta'$ then $U \cap F_{\delta'} \neq \emptyset$. In other words, $x \in \operatorname{Lim}_{\Delta}\inf F_\delta$ if and only if for every neighborhood U of x, the set

$$B(U) = \{\delta \mid \delta \in \Delta \ \& \ U \cap F_\delta = \emptyset\}$$

is not cofinal in Δ.

2.3 Definition. F is said to be convergent or to converge to F_0, $F_0 = \operatorname{Lim}_{\Delta} F_\delta$, provided $F_0 = \operatorname{Lim}_{\Delta}\inf F_\delta = \operatorname{Lim}_{\Delta}\sup F_\delta$.

By definition, the limits superior and inferior of F always exist whether F is convergent or not and they are both closed subsets of X. Among their elementary properties, the following are listed for later reference.

2.4 Lemma. (a) $\operatorname{Lim}_{\Delta}\sup F_\delta = \operatorname{Lim}_{\Delta}\sup \overline{F}_\delta = \overline{\operatorname{Lim}_{\Delta}\sup F_\delta}$

(b) $\operatorname{Lim}_{\Delta}\inf F_\delta = \operatorname{Lim}_{\Delta}\inf \overline{F}_\delta = \overline{\operatorname{Lim}_{\Delta}\inf F_\delta}$

(c) If Δ' is cofinal in Δ, then

$$\operatorname{Lim}_{\Delta}\inf F_\delta \subset \operatorname{Lim}_{\Delta'}\inf F_\delta \subset \operatorname{Lim}_{\Delta'}\sup F_\delta \subset \operatorname{Lim}_{\Delta}\sup F_\delta$$

(d) If K is a compact subset of X and the set

$A(K) = \{\delta \mid \delta \varepsilon \Delta \ \& \ K \cap F_\delta \neq \emptyset\}$ is cofinal in Δ, then $K \cap \operatorname{Lim}_\Delta \sup F_\delta \neq \emptyset$.

(e) If U is an open subset of X and the set $B(U) = \{\delta \mid \delta \varepsilon \Delta \ \& \ U \cap F_\delta = \emptyset\}$ is cofinal in Δ, then $U \cap \operatorname{Lim}_\Delta \inf F_\delta = \emptyset$.

By (a) and (b) of the lemma above, the investigation of these limit operations in $\mathfrak{A}(X)$ quickly narrows down to that of these operations in the set 2^X of closed subsets of X. Relative to functions $F: \Delta \to 2^X$, we have

2.5 Lemma. (a) If $F_\delta = F_0$ for each $\delta \varepsilon \Delta$, then $F_0 = \operatorname{Lim}_\Delta F_\delta$.

(b) If $F_0 = \operatorname{Lim}_\Delta F_\delta$ and Δ' is cofinal in Δ, then $F_0 = \operatorname{Lim}_{\Delta'} F_\delta$.

(c) If $F_0 = \operatorname{Lim}_\Delta F_\delta$ and $F_0' = \operatorname{Lim}_\Delta F_\delta$, then $F_0 = F_0'$.

Now suppose for any set $\mathfrak{F} \subset 2^X$, we tentatively define the "closure" of $\mathfrak{F}$, $\overline{\mathfrak{F}}$, to be the set of all closed sets $F_0 \varepsilon 2^X$ such that there exists a directed set Δ and a function $F: \Delta \to \mathfrak{F}$ with $F_0 = \operatorname{Lim}_\Delta F_\delta$. We do not thereby obtain a legitimate closure operator, for in general $\overline{\overline{\mathfrak{F}}} \neq \overline{\mathfrak{F}}$. (For an example, see Kuratowski [10, p. 248].) We now turn our attention to a topology for 2^X which under certain circumstances reproduces the same limit operation.

Let K be a closed compact subset of a topological space X, and $\mathfrak{U}$ a finite collection of non-empty open subsets of X. The empty set is considered compact. The collection $\mathfrak{U}$ itself may or may not be empty. Let $\mathfrak{F}(K,\mathfrak{U})$ then denote the set of all closed subsets $F \subset X$ such that (a) $K \cap F = \emptyset$, and (b) $U \cap F \neq \emptyset$ for every $U \varepsilon \mathfrak{U}$.

If $\mathfrak{U}$ is the empty collection, we shall denote it by $\emptyset$ and write $\mathfrak{F}(K,\emptyset)$ to indicate the set of closed sets satisfying condition (a); if $\mathfrak{U}$ consists of a single non-empty open set U, we shall write $\mathfrak{F}(K,\{U\})$ for $\mathfrak{F}(K,\mathfrak{U})$. If

K is the closed compact set whose sole member is a point x, we shall write $\mathfrak{F}(\{x\},\mathfrak{U})$ for $\mathfrak{F}(K,\mathfrak{U})$.

As immediate consequences of our notational conventions, we have the following identities:

2.6 (a) $2^X = \mathfrak{F}(\emptyset,\emptyset)$

(b) $\mathfrak{F}(K\cup K',\mathfrak{U}\cup\mathfrak{U}') = \mathfrak{F}(K,\mathfrak{U})\cap\mathfrak{F}(K',\mathfrak{U}')$

By 2.6, the aggregate of sets $\mathfrak{F}(K,\mathfrak{U})$, which is generated as K range over the closed compact sets of X and $\mathfrak{U}$ range over the finite collections of non-empty open sets of X, is an open basis for a topology for 2^X. The resulting topology will be designated as the p-topology for 2^X, and the topological space, 2^X together with the p-topology, will be indicated by $(2^X,p)$ or 2^X_p.

If the set K occurring in the definition of $\mathfrak{F}(K,\mathfrak{U})$ is merely required to be closed and not necessarily compact, the reulting topology is the Vietoris finite topology for 2^X (see E. Michael [12], or O. Frink [8]). In this case the element $\emptyset \in 2^X$ is always isolated. The finite topology is equivalent to the p-topology if X is compact and only if X is compact.

2.7 <u>Theorem</u>. (a) Every $(2^X,p)$ is a T_0-space.

(b) If X is a T_1-space, then $(2^X,p)$ is a T_1-space and X is imbedded homeomorphically in $(2^X,p)$ under the mapping $x \longrightarrow \{x\}$.

Proof. (a) Suppose $F_1 \neq F_2$ and there is a point x say in F_1 which is not in F_2. Let $U = X - F_2$. Then $F_1 \in \mathfrak{F}(\emptyset,\{U\})$ and $F_2 \notin \mathfrak{F}(\emptyset,\{U\})$.

(b) Suppose $F_1 \neq F_2$ and $x \in F_1 - F_2$ as in part (a) of this proof. Again let $U = X - F_2$. Then

$$F_1 \in \mathfrak{F}(\emptyset,\{U\}) \text{ and } F_2 \notin \mathfrak{F}(\emptyset,\{U\})$$

$$F_2 \in \mathfrak{F}(\{x\},\emptyset) \text{ and } F_1 \notin \mathfrak{F}(\{x\},\emptyset)$$

The second conclusion of part (b) follows easily from the restriction that the compact set K defining $\mathfrak{F}(K,\mathcal{U})$ be also closed. q.e.d.

Remark. Let X be a Hausdorff space and let $\eta: X \to 2^X$ be the natural mapping, $\eta(x) = \{x\}$. If $\eta(X) = \mathcal{X}$, then $\mathcal{X} \cup \{\emptyset\}$ is closed in 2^X_p, and in general if $\mathcal{X}(n)$ denote the family of closed subsets consisting of at most n points, $\mathcal{X}(n)$ is closed in 2^X_p. Hence the family $\mathcal{X}_f$ of finite subsets of a Hausdorff space is a F_σ in 2^X_p. Moreover $\mathcal{X}_f$ is dense in 2^X_p.

The next theorem serves as a preliminary step to Theorem 2.9 in which we prove that every $(2^X,p)$ is compact. The machinery of directed systems and convergence proves useful in this connection.

2.8 Theorem. If $F_0 = \underset{\Delta}{\mathrm{Lim}}\, F_\delta$, then $F_0 = \underset{\Delta}{\lim}\, F_\delta$ in the p-topology.

Proof. The proof, by contradiction, is modeled after that of Alexandroff and Hopf [1, pp. 114-115].

Since $F_0 = \underset{\Delta}{\mathrm{Lim}}\, F_\delta$ is closed, it is a member of 2^X. Now suppose the function $F: \Delta \to 2^X$ does not converge to F_0 in 2^X_p. Then there is a basic set $\mathfrak{F}(K,\mathcal{U})$ such that $F_0 \in \mathfrak{F}(K,\mathcal{U})$ and $\Delta' = F^{-1}(2^X - \mathfrak{F}(K,\mathcal{U}))$ is cofinal in Δ.

$$\Delta' = \{\delta \mid K \cap F_\delta \neq \emptyset\} \cup \bigcup_{i=1}^{n} \{\delta \mid U_i \cap F_\delta = \emptyset,\ U_i \in \mathcal{U}\}$$

Then one of the sets occurring in the above union is cofinal in Δ. Suppose $\{\delta \mid K \cap F_\delta \neq \emptyset\}$ is cofinal. Then $K \cap \mathrm{Lim}_\Delta \sup F_\delta \neq \emptyset$. But by hypothesis, $\mathrm{Lim}_\Delta \sup F_\delta = \underset{\Delta}{\mathrm{Lim}}\, F_\delta = F_0 \in \mathfrak{F}(K,\mathcal{U})$. Hence $K \cap \mathrm{Lim}_\Delta \sup F_\delta = \emptyset$. Hence $\{\delta \mid K \cap F_\delta \neq \emptyset\}$ is not cofinal and $\{\delta \mid U_i \cap F_\delta = \emptyset\}$ for some $U_i \in \mathcal{U}$ must be cofinal. But in this case $U_i \cap \mathrm{Lim}_\Delta \inf F_\delta = \emptyset$. Again by hypothesis $\mathrm{Lim}_\Delta \inf F_\delta = \underset{\Delta}{\mathrm{Lim}}\, F_\delta = F_0 \in \mathfrak{F}(K,\mathcal{U})$. Hence $U_i \cap \mathrm{Lim}_\Delta \inf F_\delta \neq \emptyset$. Our assumption that F does not converge to F_0 in 2^X_p has lead to a contradiction. q.e.d.

2.9 Theorem. Every $(2^X,p)$ is compact.

Proof. Let F be an ultrafunction on a directed system Δ into $(2^X,p)$. Then for every open set $\mathcal{G} \subset 2^X$, either $F^{-1}(\mathcal{G})$ or $F^{-1}(2^X - \mathcal{G})$ is not cofinal in Δ. In particular, for open sets of the type $\mathfrak{F}(\emptyset,\{U\})$, either $F^{-1}(\mathfrak{F}(\emptyset,\{U\}))$ or $F^{-1}(2^X - \mathfrak{F}(\emptyset,\{U\}))$ is not cofinal in Δ. If $\operatorname{Lim}_\Delta\sup F_\delta = \emptyset$, then $\emptyset = \operatorname{Lim}_\Delta F_\delta$ and by Theorem 2.8, F converges to $\emptyset$ in 2^X_p.

Now suppose $x_0 \varepsilon \operatorname{Lim}_\Delta\sup F_\delta$. Let U_0 be an arbitrary open neighborhood of x_0. Then by definition, the set $\Delta' = \{\delta \mid U_0 \cap F_\delta \neq \emptyset\}$ is cofinal in Δ. But $\Delta' = F^{-1}(\mathfrak{F}(\emptyset,\{U_0\}))$ and hence $\Delta'' = F^{-1}(2^X - \mathfrak{F}(\emptyset,\{U_0\}))$ is not cofinal in Δ. But $\Delta'' = \{\delta \mid U_0 \cap F_\delta = \emptyset\}$. Since the neighborhood U_0 of x_0 was arbitrarily chosen, by definition $x_0 \varepsilon \operatorname{Lim}_\Delta\inf F_\delta$. Hence $\operatorname{Lim}_\Delta\sup F_\delta \subset \operatorname{Lim}_\Delta\inf F_\delta$ and $F_0 = \operatorname{Lim}_\Delta F_\delta$ exists. Again by Theorem 2.8, F converges to F_0 in 2^X_p.

It has been shown that every ultrafunction in $(2^X,p)$ is convergent. Therefore by Theorem 1.6 (b), $(2^X,p)$ is compact.

The next theorem has been demonstrated by G. Choquet [6] for the set of non-empty closed subsets of a space.

2.10 <u>Theorem.</u> Let X be a Hausdorff space. Then each of the following conditions is equivalent to local compactness of X:

(a) $F_0 = \operatorname{Lim}_\Delta F_\delta$ if and only if $F_0 = \lim_\Delta F_\delta$ in the p-topology.

(b) $(2^X,p)$ is a Hausdorff space.

Proof. 1) Local compactness of X implies (a). By Theorem 2.8, if $F_0 = \operatorname{Lim}_\Delta F_\delta$, then $F_0 = \lim_\Delta F_\delta$ in the p-topology whether X is locally compact or not. Now suppose $F: \Delta \rightarrow 2^X_p$ converges to F_0, i.e. $F_0 = \lim_\Delta F_\delta$ in the p-topology.

$F_0 \subset \operatorname{Lim}_\Delta\inf F_\delta$. For suppose $x \varepsilon F_0$ and $U(x)$ is an arbitrary neighborhood of x. $F_0 \varepsilon \mathfrak{F}(\emptyset,\{U(x)\})$. Then $\Delta' = F^{-1}(2^X - \mathfrak{F}(\emptyset,\{U(x)\}))$ is not cofinal in Δ. But $\Delta' = \{\delta \mid U(x) \cap F_\delta = \emptyset\}$. Hence $x \in \mathrm{Lim}_\Delta \inf F_\delta$.

$\mathrm{Lim}_\Delta \sup F_\delta \subset F_0$. Suppose $x \notin F_0$. Then there is a neighborhood $U(x)$ of x such that $\overline{U(x)}$ is compact and $\overline{U(x)} \cap F_0 = \emptyset$. Then $F_0 \in \mathfrak{F}(\overline{U(x)}, \emptyset)$ and $\Delta' = F^{-1}(2^X - \mathfrak{F}(\overline{U(x)}, \emptyset))$ is not cofinal in Δ. $\Delta' = \{\delta \mid \overline{U(x)} \cap F_\delta \neq \emptyset\}$ Hence $\Delta'' = \{\delta \mid U(x) \cap F_\delta \neq \emptyset\} \subset \Delta'$ is not cofinal and $x \notin \mathrm{Lim}_\Delta \sup F_\delta$. Thus $\mathrm{Lim}_\Delta \sup F_\delta \subset F_0 \subset \mathrm{Lim}_\Delta \inf F_\delta$, and $\mathrm{Lim}_\Delta F_\delta$ exists and is equal to F_0.

2) (a) implies (b). This is a consequence of Lemma 2.5 (c).

3) (b) implies X is locally compact. Let $\mathfrak{X}$ be the set of all closed subsets of X each of which consists of a single point. X is homeomorphic to $\mathfrak{X}$. $\mathfrak{X} \cup \{\emptyset\}$ is closed in $(2^X, p)$ as remarked previously. $(2^X, p)$ is a Hausdorff space by hypothesis and moreover compact by Theorem 2.9. But a subspace of a locally compact Hausdorff space which is the difference of two closed sets is itself a locally compact space. $\mathfrak{X}$ is such a subspace of $(2^X, p)$. Hence X is locally compact. q.e.d.

2.11 <u>Theorem.</u> If X is a locally compact Hausdorff space and has a basis $\mathcal{B}$ of cardinal number b, then $(2^X, p)$ has a basis of cardinal number not greater than b (2^b if b is finite).

Proof. The finite case is trivial. As for the non-finite case, we may consider open sets $U \in \mathcal{B}$ to have compact closures. Let $\mathfrak{M}$ be the collection of all finite collections of non-empty open sets in $\mathcal{B}$. $\mathfrak{M}$ has cardinal number $C(\mathfrak{M}) = C(\mathcal{B}) = b$. Let $\mathcal{K}$ be the collection of compact sets which are unions of closures of a finite number of open sets in $\mathcal{B}$. $C(\mathcal{K}) = b$. Then the set of all pairs $(K, \mathfrak{U})$ such that $K \in \mathcal{K}$ and $\mathfrak{U} \in \mathfrak{M}$ has cardinal number equal to b. It is easy to see that a subaggregate of the aggregate $\{\mathfrak{F}(K, \mathfrak{U}) \mid K \in \mathcal{K}, \mathfrak{U} \in \mathfrak{M}\}$ is a basis for $(2^X, p)$. q.e.d.

<u>Corollary</u>. Let X be a Hausdorff space. Then $(2^X,p)$ is metrizable if and only if X is a perfectly separable locally compact space.

Proof. Necessity. If $(2^X,p)$ is metrizable, then $(2^X,p)$ is perfectly separable since it is compact. Then X is perfectly separable since X is imbedded in $(2^X,p)$. The metrizability of $(2^X,p)$ also implies that $(2^X,p)$ is a Hausdorff space and therefore by Theorem 2.10, X is locally compact.

Sufficiency. If X is a perfectly separable locally compact Hausdorff space, then by the above theorem, $(2^X,p)$ is perfectly separable. Moreover, $(2^X,p)$ is a compact Hausdorff space, hence metrizable.

As an application of the p-topology, a theorem on interior or open maps is interposed at this point. This theorem will in turn motivate the definition of interior convergence.

2.12 <u>Definition</u>. A continuous function $f\colon X \to Y$ is said to be <u>interior</u> or <u>open</u> if the image $f(U)$ of every open subset U of X is open relative to $f(X)$. f is <u>strongly interior</u> or <u>strongly open</u> if $f(U)$ is open in Y.

2.13 <u>Theorem</u>. Let $f\colon X \to Y$ be a continuous function on a topological space X into a Hausdorff space Y. Define a transformation $f^{-1}\colon Y \to 2^X$ by $f^{-1}(y) = \{x \mid y = f(x)\}$. Then each of the following conditions is equivalent to strong interiority of f:

(a) If $y = \lim_{\Delta} y_\delta$, then $f^{-1}(y) = \operatorname{Lim}_{\Delta} f^{-1}(y_\delta)$.

(b) If $y = \lim_{\Delta} y_\delta$, then $f^{-1}(y) = \lim_{\Delta} f^{-1}(y_\delta)$ in the p-topology.

(c) f^{-1} is continuous on Y into $(2^X,p)$

Proof. 1) Let f be strongly interior and suppose $y = \lim_{\Delta} y_\delta$. To show that $\operatorname{Lim\,sup}_{\Delta} f^{-1}(y_\delta) \subset f^{-1}(y)$, we require only the continuity of f. Suppose

$x \notin f^{-1}(y)$. Then there are neighborhoods V of $f(x)$ and W of y such that $V \cap W = \emptyset$. Since f is continuous, there is a neighborhood U of x such that $f(U) \subset V$, and so by the choice of V and W, $U \cap f^{-1}(W) = \emptyset$. Since $y = \lim_{\Delta} y_\delta$, the set $\Delta' = \{\delta \mid y_\delta \notin W\}$ is not cofinal in Δ. Hence $\Delta'' = \{\delta \mid U \cap f^{-1}(y_\delta) \neq \emptyset\}$ which is contained in Δ', is not cofinal in Δ, and $x \notin \mathrm{Lim}_{\Delta}\sup f^{-1}(y_\delta)$.

$f^{-1}(y) \subset \mathrm{Lim}_{\Delta}\inf f^{-1}(y_\delta)$. Let $x \in f^{-1}(y)$ and U be an arbitrary neighborhood of x. Since f is strongly interior, f(U) is open in Y and since $y = f(x) = \lim_{\Delta} y_\delta$, the set $\Delta_o = \{\delta \mid y_\delta \notin f(U)\}$ is not cofinal in Δ. But $\Delta_o = \{\delta \mid U \cap f^{-1}(y_\delta) = \emptyset\}$. Thus $x \in \mathrm{Lim}_{\Delta}\inf f^{-1}(y_\delta)$.

2) (a) implies (b). This is a consequence of Theorem 2.8

3) (b) is equivalent to (c).

4) (c) implies the strong interiority of f. Suppose on the contrary, f is not strongly interior. Then there is an open set U of X such that f(U) is not open in Y. Then there is a point $y = f(x) \in f(U)$ such that for every neighborhood V of y, there is a point $y' \in V$ which is not in f(U), i.e. $U \cap f^{-1}(y') = \emptyset$. But $f^{-1}(y) \in \mathfrak{F}(\emptyset, \{U\})$ and $U \cap f^{-1}(y') = \emptyset$ implies $f^{-1}(y')$ does not belong to $\mathfrak{F}(\emptyset, \{U\})$. Hence f^{-1} is not continuous at y. q.e.d.

Theorem 2.13 is not new. For the equivalence of strong interiority and (a), see Choquet [6]. For the equivalence of strong interiority and (b) or (c), see Whyburn [15, p. 18].

In the course of the above proof, we have employed the same symbol f^{-1} to denote two distinct relations. For as f^{-1} is defined, if $V \subset Y$, then $f^{-1}(V)$ is a subset of 2^X, but we have also taken $f^{-1}(V)$ as the subset $\bigcup_{y \in V} f^{-1}(y)$ of X. But in context, it is usually clear which of the two meanings is intended and in the sequel we shall continue to permit ourselves this latitude in notation when there is no danger of confusion.

Observe that as far as inverse images of a strongly interior map f are concerned, by the above theorem, $\underset{\Delta}{\mathrm{Lim}}\, f^{-1}(y_\delta)$ and $\underset{\Delta}{\lim}\, f^{-1}(y_\delta)$ in the p-topology have the same meaning in contradistinction to Theorem 2.10 which requires that X be locally compact in order that this holds.

A set of criteria for interiority, not necessarily strong interiority, similar to those of Theorem 2.13 may of course be given for mappings f, by restricting the transformation f^{-1} to the range f(X) of f, or in another sense to $\overline{f(X)}$. From this point of view, Theorem 2.13 is a generalization of the well-known Eilenberg's criterion for interiority of mappings in metric spaces (Eilenberg [7], Kuratowski [11, p. 48]). For later reference, a formal statement of this generalization is given here.

2.14 Eilenberg's Theorem. A continuous function $f: X \to Y$ on a space X into a Hausdorff space Y is interior if and only if whenever $y = \underset{\Delta}{\lim}\, y_\delta$, where $y_\delta \in f(X)$ for each $\delta \in \Delta$, $f^{-1}(y) = \underset{\Delta}{\mathrm{Lim}}\, f^{-1}(y_\delta)$.

Proof. Necessity. To prove the necessity, consider the two cases (a) $y = \underset{\Delta}{\lim}\, y_\delta \in f(X)$, and (b) $y = \underset{\Delta}{\lim}\, y_\delta \notin f(X)$. If (a) is the case, then the conclusion $f^{-1}(y) = \underset{\Delta}{\mathrm{Lim}}\, f^{-1}(y_\delta)$ follows from Theorem 2.13 by letting the space Y appearing in the statement of that theorem be equal to f(X) and noting that a function $f: X \to Y$ is interior if and only if $f: X \to f(X)$ is strongly interior. On the other hand, in case (b), $f^{-1}(y) = \emptyset$. But by the continuity of f, $\underset{\Delta}{\mathrm{Lim\,sup}}\, f^{-1}(y_\delta) \subset f^{-1}(y)$ as shown in the first part of 1) in the proof of Theorem 2.13. Hence $\underset{\Delta}{\mathrm{Lim}}\, f^{-1}(y_\delta)$ exists and is equal to $\emptyset$.

Sufficiency. By Theorem 2.13, considering only the case where $y = \underset{\Delta}{\lim}\, y_\delta \in f(X)$, the function $f: X \to f(X)$ is strongly interior. Hence $f: X \to Y$ is an interior function. q.e.d.

The transformation f^{-1} restricted to f(X), is not only continuous but also a homeomorphism of $f(X)$ onto a subspace $\mathcal{D}$ of $(2^X, p)$, where $\mathcal{D}$ is the disjoint decomposition of X into the closed sets $f^{-1}(y)$, $y \varepsilon f(X)$. The continuity of $(f^{-1})^{-1}: \mathcal{D} \to f(X)$ follows directly from the continuity of f and the definition of the p-topology.

III. INTERIOR CONVERGENCE

The definition of continuous convergence (1.7) and Eilenberg's theorem suggest a type of convergence based on the convergence property mentioned in the Introduction which relates the compact-open and g-topologies. Let $f\colon \Delta \rightarrow T(X,Y)$ be a function on a directed system Δ into a set of transformations $T(X,Y)$ on a space X into another Y.

3.1 <u>Definition</u>. f is said to <u>converge interiorly</u> to $f_0 \in T(X,Y)$, $f_\delta \rightarrow f_0$ interiorly, provided,

(a) $f_\delta \rightarrow f_0$ continuously and

(b) for every directed system Σ and function $y\colon \Delta \times \Sigma \rightarrow Y$, such that $y_{\delta\sigma} \in f_\delta(X)$, if y is convergent, then $\operatorname{Lim}_{\Delta\times\Sigma} f_\delta^{-1}(y_{\delta\sigma})$ exists.

The concept of interior convergence of a sequence of functions is due to M. K. Fort, Jr. The definition given above for directed sets of functions has been suggested by analogy. It is desirable to ascertain that interior convergence thus defined, when applied to a suitably restricted class of functions, possesses some of the properties which are usually ascribed to convergence systems. For this and other purposes, it is convenient to establish the following lemma.

3.2 <u>Lemma</u>. If $f_\delta \rightarrow f_0$ continuously, where f_δ, f_0 are transformations on a space X into a Hausdorff space Y and $y_0 = \lim_{M} y_\mu$, then $\operatorname{Lim\,sup}_{\Delta\times M} f_\delta^{-1}(y_\mu) \subset f_0^{-1}(y_0)$.

Proof. Suppose $x \notin f_0^{-1}(y_0)$. Then there are neighborhoods V of $f_0(x)$ and W of y_0 such that $V \cap W = \emptyset$. Since $f_\delta \rightarrow f_0$ continuously, there is a neighborhood U of x and a $\delta' \varepsilon \Delta$ such that $f_\delta(U) \subset V$ whenever $\delta > \delta'$. $y_0 = \lim_{M} y_\mu \varepsilon W$ and $V \cap W = \emptyset$. Hence the set $A(U) = \{(\delta,\mu) \mid U \cap f_\delta^{-1}(y_\mu) \neq \emptyset\}$ is not cofinal in $\Delta \times M$ and $x \notin \operatorname{Lim\,sup}_{\Delta \times M} f_\delta^{-1}(y_\mu)$. q.e.d.

3.3 Theorem. Let I(X,Y) be the set of interior mappings on a space X into a Hausdorff space Y and let $f: \Delta \to I(X,Y)$ be a function on a directed system Δ into I(X,Y).

(a) If $f_\delta = f_0$ for each $\delta \varepsilon \Delta$, then $f_\delta \to f_0$ interiorly.

(b) If $f_\delta \xrightarrow[\Delta]{} f_0$ interiorly and Δ_1 is cofinal in Δ, then $f_\delta \xrightarrow[\Delta_1]{} f_0$ interiorly.

(c) If $f_\delta \to f_0$ and $f_\delta \to f_0'$ interiorly, then $f_0 = f_0'$.

Proof. (a) is Eilenberg's theorem.

(b) If $f_\delta \xrightarrow[\Delta]{} f_0$ continuously, then $f_\delta \xrightarrow[\Delta_1]{} f_0$ continuously, so part (a) of Definition 3.1 is satisfied. Now suppose $y_0 = \lim_{\Delta_1 \times \Sigma} y_{\delta\sigma}$, $y_{\delta\sigma} \varepsilon f_\delta(X)$ for every $\delta \varepsilon \Delta_1$. The diagonal D of $\Delta_1 \times \Delta_1$ is cofinal in $\Delta_1 \times \Delta_1$, and hence $D \times \Sigma$ is cofinal in $\Delta_1 \times \Delta_1 \times \Sigma$. $\operatorname{Lim\,sup}_{D \times \Sigma} f_\delta^{-1}(y_{\delta\sigma}) = \operatorname{Lim\,sup}_{\Delta_1 \times \Sigma} f_\delta^{-1}(y_{\delta\sigma})$ is contained in $\operatorname{Lim\,sup}_{\Delta_1 \times \Delta_1 \times \Sigma} f_\delta^{-1}(y_{\delta'\sigma})$ so by Lemma 3.2, $\operatorname{Lim\,sup}_{\Delta_1 \times \Sigma} f_\delta^{-1}(y_{\delta\sigma}) \subset f_0^{-1}(y_0)$. Therefore if $f_0^{-1}(y_0) = \emptyset$, then $\operatorname{Lim}_{\Delta_1 \times \Sigma} f_\delta^{-1}(y_{\delta\sigma})$ exists and is equal to $\emptyset$. Suppose $f_0^{-1}(y_0) \neq \emptyset$. Choose an $x_0 \varepsilon f_0^{-1}(y_0)$ and for every $\delta \notin \Delta_1$ and $\sigma \varepsilon \Sigma$ define $y_{\delta\sigma} = f_\delta(x_0)$. Then $y_0 = \lim_{\Delta \times \Sigma} y_{\delta\sigma} = \lim_{\Delta_1 \times \Sigma} y_{\delta\sigma}$ and since $f_\delta \to f_0$ interiorly on Δ, $\operatorname{Lim}_{\Delta \times \Sigma} f_\delta^{-1}(y_{\delta\sigma})$ exists. But $\Delta_1 \times \Sigma$ is cofinal in $\Delta \times \Sigma$. Hence $\operatorname{Lim}_{\Delta_1 \times \Sigma} f_\delta^{-1}(y_{\delta\sigma})$ exists.

(c) follows from the hypothesis that Y is a Hausdorff space. q.e.d

By the above theorem, interior convergence in I(X,Y) enjoys the same convergence properties as those enunciated in Lemma 2.5 for the Painlevé limit operation in the class of closed subsets of a topological space. As in the case of the Painlevé limit operation, we may again attempt to define a "closure" operator in $I(X,Y)$, here in terms of interior convergence. (See remark following Lemma 2.5.) The condition, $\bar{\bar{A}} = \bar{A}$ for $A \subset I(X,Y)$, again fails in general. This will be shown by an example after the next theorem.

3.4 Theorem. If $f_\delta \to f_0$ interiorly, where f_δ, f_0 are transformations on a space X into a Hausdorff space Y, and if $y_0 = \lim_{\Delta\times\Sigma} y_{\delta\sigma}$ where $y_{\delta\sigma} \varepsilon f_\delta(X)$ for each $\delta \varepsilon \Delta$, then $f_0^{-1}(y_0) = \operatorname{Lim}_{\Delta\times\Sigma} f_\delta^{-1}(y_{\delta\sigma})$.

Proof. Since the convergence is interior, $\operatorname{Lim}_{\Delta\times\Sigma} f_\delta^{-1}(y_{\delta\sigma})$ exists, and by Lemma 3.2, $\operatorname{Lim}_{\Delta\times\Sigma} f_\delta^{-1}(y_{\delta\sigma}) = \operatorname{Lim\,sup}_{\Delta\times\Sigma} f_\delta^{-1}(y_{\delta\sigma}) \subset f_0^{-1}(y_0)$.

To show that $f_0^{-1}(y_0) \subset \operatorname{Lim}_{\Delta\times\Sigma} f_\delta^{-1}(y_{\delta\sigma})$, we begin by inserting here a remark on directed systems. Let A and A' be two disjoint directed systems which are isomorphic, i.e. there is a one-to-one correspondence $\theta: A \to A'$ which takes A onto A' and which preserves the directing relations in A and A': $\alpha_1 < \alpha_2$ in A if and only if $\theta(\alpha_1) \overset{'}{<} \theta(\alpha_2)$ in A'. Define a binary relation $\overset{*}{<}$ in their union $A^* = A + A'$ by letting $\overset{*}{<}$ have the same meaning as $<$ for each pair $\alpha_1, \alpha_2 \varepsilon A$, and the same meaning as $\overset{'}{<}$ for $\theta(\alpha_1), \theta(\alpha_2) \varepsilon A'$. For a mixed pair, $\alpha_1^* = \alpha_1 \varepsilon A$ $(\alpha_1^* = \theta(\alpha_1))$ and $\alpha_2^* = \theta(\alpha_2) \varepsilon A'$ $(\alpha_2^* = \alpha_2 \varepsilon A)$, let $\alpha_1^* \overset{*}{<} \alpha_2^*$ if and only if $\alpha_1 < \alpha_2$ in A. Then A^* is directed by $\overset{'}{<}$, and A and A' are disjoint cofinal sets in A^*. If B is a third directed system, $B \times (A + A') = (B \times A) + (B \times A')$, where the isomorphism $\zeta: B \times A \to B \times A'$ is the obvious one: $\zeta(\beta,\alpha) = (\beta, \theta(\alpha))$. $B \times A$ and $B \times A'$ are disjoint cofinal sets in $B \times (A + A')$.

Now take the directed system Σ which occurs in the statement of the theorem, distinguish each $\sigma \varepsilon \Sigma$ by a prime symbol ' , and form a replica Σ' of Σ, where $\sigma_1' < \sigma_2'$ in Σ' if and only if $\sigma_1 < \sigma_2$ in Σ. Σ and Σ' are considered disjoint.

Suppose $x_0 \varepsilon f_0^{-1}(y_0)$. Define $y': \Delta \times \Sigma' \to Y$ by $y'_{\delta\sigma'} = f_\delta(x_0)$, independent of $\sigma' \varepsilon \Sigma'$. Then $y_0 = f_0(x_0) = \lim_{\Delta\times\Sigma'} y'_{\delta\sigma'}$. By the interiority of convergence, $\operatorname{Lim}_{\Delta\times\Sigma'} f_\delta^{-1}(y'_{\delta\sigma'})$ exists and here $x_0 \varepsilon \operatorname{Lim}_{\Delta\times\Sigma'} f_\delta^{-1}(y'_{\delta\sigma'})$. Now let $\Sigma^* = \Sigma + \Sigma'$ and consider the function y^* on $\Delta\times\Sigma^* = (\Delta\times\Sigma)+(\Delta\times\Sigma')$ into Y defined by

$$y^*_{\delta\sigma^*} = \begin{cases} y_{\delta\sigma} & \text{if } \sigma^* = \sigma \varepsilon \Sigma \\ y'_{\delta\sigma'} & \text{if } \sigma^* = \sigma' \varepsilon \Sigma' \end{cases}$$

$y_o = \lim_{\Delta\times\Sigma^*} y^*_{\delta\sigma^*} = \lim_{\Delta\times\Sigma} y_{\delta\sigma} = \lim_{\Delta\times\Sigma'} y'_{\delta\sigma'}$ and $y^*_{\delta\sigma^*} \varepsilon f_\delta(X)$ for each $\delta \varepsilon \Delta$. Hence $\operatorname{Lim}_{\Delta\times\Sigma^*} f_\delta^{-1}(y^*_{\delta\sigma^*})$ exists. But $\Delta\times\Sigma$ and $\Delta\times\Sigma'$ are cofinal in $\Delta\times\Sigma^*$. Hence $\operatorname{Lim}_{\Delta\times\Sigma^*} f_\delta^{-1}(y^*_{\delta\sigma^*}) = \operatorname{Lim}_{\Delta\times\Sigma} f_\delta^{-1}(y_{\delta\sigma}) = \operatorname{Lim}_{\Delta\times\Sigma'} f_\delta^{-1}(y'_{\delta\sigma'})$ and $x_0 \varepsilon \operatorname{Lim}_{\Delta\times\Sigma} f_\delta^{-1}(y_{\delta\sigma})$. q.e.d.

<u>Example</u>. Let X be the space consisting of the real numbers 1/2, 1/3,... and 0,1,2,3,... with the usual topology. Define a sequence of functions $f_n\colon X \to X$ by

$$f_n(x) = \begin{cases} 1/n & \text{if } 0 \le x \le 1/n \\ x & \text{if } x > 1/n \end{cases}$$

Each f_n is a strongly open map and $f_n \to f_o$ interiorly, where f_o is the identity map. Now define $g_{nm}\colon X \to X$ by

$$g_{nm}(x) = \begin{cases} 1/(n+m) & \text{if } x = n+m \\ f_n(x) & \text{otherwise} \end{cases}$$

Each g_{nm} is a strongly open map and for each n, $g_{nm} \xrightarrow[m]{} f_n$ interiorly. Let $G = \{g_{nm} \mid n,m = 1,2,3,\ldots\}$. It is easy to see that there is no directed set of functions taken from G which converges interiorly to the identity f_o. Thus if $\overline{G}$ is defined to be the set of open maps g_o such that there exists a $g\colon \Delta \to G$ so that $g_\delta \to g_o$ interiorly, then in general $\overline{\overline{G}} \neq \overline{G}$.

In connection with Theorem 2.13, it has been remarked that $\operatorname{Lim}_{\Sigma} f^{-1}(y_\sigma)$ and $\lim_{\Sigma} f^{-1}(y_\sigma)$ in the p-topology are equal as far as the limits of inverse images of a single interior mapping are concerned. If we restrict ourselves to continuous functions, the equality of these limits in 2^X for inverse images of a directed set of functions which converges interiorly can also be shown. More precisely, suppose part (b) of the definition of interior convergence is replaced by

(b') If $y_0 = \lim_{\Delta\times\Sigma} y_{\delta\sigma}$, $y_{\delta\sigma} \varepsilon f_\delta(X)$, then $f_0^{-1}(y_0) = \mathrm{Lim}_{\Delta\times\Sigma} f_\delta^{-1}(y_{\delta\sigma})$, or

(b") If $y_0 = \lim_{\Delta\times\Sigma} y_{\delta\sigma}$, $y_{\delta\sigma} \varepsilon f_\delta(X)$, then $f_0^{-1}(y_0) = \lim_{\Delta\times\Sigma} f_\delta^{-1}(y_{\delta\sigma})$ in 2_p^X.

By Theorem 3.4, (a),(b) of Definition 3.1 is equivalent to (a),(b'). By Theorem 2.8, (a),(b') implies (a),(b"). Now suppose we accept (a),(b") as the definition of interior convergence. By (a), $\mathrm{Lim\,sup}_{\Delta\times\Sigma} f_\delta^{-1}(y_{\delta\sigma}) \subset f_0^{-1}(y_0)$. But in all cases, $\lim_{\Delta\times\Sigma} f_\delta^{-1}(y_{\delta\sigma}) \subset \mathrm{Lim\,inf}_{\Delta\times\Sigma} f_\delta^{-1}(y_{\delta\sigma})$, as we may readily verify for any convergent function into 2_p^X. Hence (a),(b") implies (a),(b') and is therefore equivalent to (a),(b'). (a),(b") are then necessary and sufficient conditions for interior convergence of continuous functions. For use in proofs of subsequent theorems, this criterion will be stated in such a manner so that the requirement of continuity may be relaxed.

3.5 <u>Theorem</u>. Let $f_\delta \rightarrow f_0$ continuously, where f_δ, f_0 are transformations on a space X into a Hausdorff space Y. Then in order that $f_\delta \rightarrow f_0$ interiorly, it is necessary and sufficient that for every open set U of X and point $y_0 \varepsilon f_0(U)$, there exist a neighborhood V_0 of y_0 and a $\delta_0 \varepsilon \Delta$ such that $V_0 \cap f_\delta(X) \subset f_\delta(U)$ whenever $\delta > \delta_0$.

Proof. Necessity. Suppose on the contrary there is a $y_0 \varepsilon f_0(U)$, such that for every neighborhood V of y_0 and every $\delta \varepsilon \Delta$, there is a δ', such that $\delta' > \delta$ and $V \cap f_{\delta'}(X) \not\subset f_{\delta'}(U)$. Let $\mathcal{V}$ be the set of neighborhoods of y_0. Then the set $A \subset \Delta\times\mathcal{V}$ of points (δ, V) such that $V \cap f_\delta(X) \not\subset f_\delta(U)$ is cofinal in $\Delta\times\mathcal{V}$. Let $x_0 \varepsilon U \cap f_0^{-1}(y_0)$. Define $y\colon \Delta\times\mathcal{V} \longrightarrow Y$ by

$$y(\delta,V) = \begin{cases} y_{\delta V} = f_\delta(x_0) & \text{if } (\delta,V) \not\varepsilon A \\ y_{\delta V} \,\varepsilon\, (V \cap f_\delta(X)) - f_\delta(U) & \text{if } (\delta,V)\,\varepsilon\, A \end{cases}$$

Then $y_0 = \lim_{\Delta\times\mathcal{V}} y_{\delta V}$ but $f_0^{-1}(y_0) \neq \operatorname{Lim}_{\Delta\times\mathcal{V}} f_\delta^{-1}(y_{\delta V})$ if the limit exist, for A is cofinal in $\Delta\times\mathcal{V}$ and $x_0 \not\varepsilon \operatorname{Lim\,inf}_A f_\delta^{-1}(y_{\delta V})$. Hence the convergence is not interior.

Sufficiency. Let $y_0 = \lim_{\Delta\times\Sigma} y_{\delta\sigma}$, $y_{\delta\sigma}\,\varepsilon\, f_\delta(X)$. If $y_0 \not\varepsilon f_0(X)$, $\operatorname{Lim}_{\Delta\times\Sigma} f_\delta^{-1}(y_{\delta\sigma})$ exists and is equal to $\emptyset = f_0^{-1}(y_0)$. Now suppose $y_0 \,\varepsilon\, f_0(X)$. Let $x_0 \,\varepsilon\, f_0^{-1}(y_0)$ and let U be an arbitrary neighborhood of x_0. Then $y_0 \,\varepsilon\, f_0(U)$ and by hypothesis there is a neighborhood V_0 of y_0 and a $\delta_0 \,\varepsilon\, \Delta$ such that $V_0 \cap f_\delta(X) \subset f_\delta(U)$ whenever $\delta > \delta_0$. Since $y_0 = \lim_{\Delta\times\Sigma} y_{\delta\sigma}$, there is a (δ',σ'), $\delta' > \delta_0$ such that $y_{\delta\sigma}\,\varepsilon\, V_0$ whenever $(\delta,\sigma) > (\delta',\sigma')$. But $V_0 \cap f_\delta(X) \subset f_\delta(U)$ whenever $\delta > \delta_0$. Thus $U \cap f_\delta^{-1}(y_{\delta\sigma}) \neq \emptyset$ whenever $(\delta,\sigma) > (\delta',\sigma')$. Since U was an arbitrarily chosen neighborhood of x_0, $x_0 \,\varepsilon\, \operatorname{Lim\,inf}_{\Delta\times\Sigma} f_\delta^{-1}(y_{\delta\sigma})$. Hence $f_0^{-1}(y_0) \subset \operatorname{Lim\,inf}_{\Delta\times\Sigma} f_\delta^{-1}(y_{\delta\sigma})$, and $\operatorname{Lim}_{\Delta\times\Sigma} f_\delta^{-1}(y_{\delta\sigma})$ exists. q.e.d.

Let X and Y be metric spaces and J the set of positive integers. Applying Theorem 3.5, it is easy to see that if $f_n \xrightarrow[J]{} f_0$ interiorly, then given any $\epsilon > 0$ and compact set $L \subset X$, there is a $\delta > 0$ and an integer n_0 such that for $n > n_0$ and any $x \,\varepsilon\, L$, we have $f_n(X) \cap S_\delta(f_n(x)) \subset f_n(S_\epsilon(x))$, where S_δ, S_ϵ are spherical neighborhoods. Whyburn [15, p. 20] has called sequences which satisfy the stronger condition $S_\delta(f_n(x)) \subset f_n(S_\epsilon(x))$, <u>uniformly approximately interior</u> on compact sets. If $f_n \rightarrow f_0$ continuously and $\{f_n\}$ is uniformly approximately interior on compact sets, then f_0 is strongly interior and $f_n \rightarrow f_0$ interiorly.

The next theorem, together with Theorem 3.3 motivate the choice of the designation "interior convergence". Let us return to continuous convergence for a moment. If $f_\delta \rightarrow f_0$ continuously, to show that the limit transformation

f_o: $X \to Y$ is continuous, by Theorem 1.8 it is sufficient to assume that Y is a regular space, the continuity of f_o in no way depending on the continuity of the functions f_δ. Now if $f_\delta \to f_o$ interiorly, in a sense, the transformations f_δ^{-1} converge continuously to f_o^{-1}. But since we have not assumed that the functions f_δ are continuous, the inverse image $f_\delta^{-1}(y)$ of a point $y \in Y$ may not be a closed subset of X and hence f_δ^{-1} may not be a transformation on Y into 2^X. We can of course remedy this defect by defining f^{-1} by $f^{-1}(y) = \overline{\{x \mid y = f\ (x)\}}$. In any event, the foregoing considerations suggest that if 2^X is a regular space under the p-topology, that is, if X is a locally compact Hausdorff space, then f_o^{-1} is continuous on $f_o(X)$ into 2^X, or f_o is an interior function if in addition f_o is continuous.

3.6 Theorem. If f converges interiorly to a continuous function f_o, where $f(\delta) = f_\delta$, f_o are transformations on a locally compact Hausdorff space X into a Hausdorff space Y, then f_o is an interior mapping.

Proof. To show that f_o is interior, we must show that for every open set $U \subset X$ and $y_o \in f_o(U)$, there is a neighborhood V of y_o such that $V \cap f_o(X) \subset f_o(U)$, i.e. $f_o(U)$ is open relative to $f_o(X)$. Choose an x_o in $U \cap f_o^{-1}(y_o)$ and a neighborhood U_o of x_o such that $x_o \in U_o \subset \overline{U}_o \subset U$ and $\overline{U}_o$ is compact. By Theorem 3.5, there is a neighborhood V_o of $y_o = f_o(x_o)$ and a $\delta_o \in \Delta$ such that $V_o \cap f_\delta(X) \subset f_\delta(U_o)$ whenever $\delta > \delta_o$. Now suppose y' is in $V_o \cap f_o(X)$. Select any point $x' \in f_o^{-1}(y')$ and define $y'_\delta = f_\delta(x')$. Then $y' = \lim_\Delta y'_\delta$. Since $y' \in V_o$, there is a δ_1 such that $y'_\delta = f_\delta(x') \in V_o$ whenever $\delta > \delta_1$. But since $V_o \cap f_\delta(X) \subset f_\delta(U_o)$ when $\delta > \delta_o$, then for $\delta > \delta_o$ and $\delta > \delta_1$, $y'_\delta \in f_\delta(U_o)$, i.e. $U_o \cap f_\delta^{-1}(y'_\delta) \neq \emptyset$. The convergence of f is continuous: hence $\operatorname{Lim}_\Delta \sup f_\delta^{-1}(y'_\delta) \subset f_o^{-1}(y')$. But $\overline{U}_o$ is compact. Therefore $\emptyset \neq \overline{U}_o \cap \operatorname{Lim}_\Delta \sup f_\delta^{-1}(y'_\delta) \subset \overline{U}_o \cap f_o^{-1}(y')$ and $y' \in f_o(\overline{U}_o)$. We have shown that

$V_0 \cap f_0(X) \subset f_0(\overline{U}_0) \subset f_0(U)$ and therefore V_0 is a neighborhood of y_0 with the desired property. q.e.d.

Theorems 3.3 and 3.6 indicate that interior convergence is fairly well-behaved within the set $I(X,Y)$ of interior mappings of a space X into a Hausdorff space Y. The question which arises now is whether or not there is a topology for $I(X,Y)$ or for any subset of $I(X,Y)$, which is consistent with our notion of interior convergence. If we cull from $I(X,Y)$ the subset $I^*(X,Y)$ of all strongly interior mappings, with appropiate conditions on the spaces X and Y, an affirmative answer to the question posed may be given. Theorem 3.5 suggests the following topology. If K and U are subsets of X, and L and V are subsets of Y, let

$M(K,V)$ be the set of functions $f \varepsilon I^*(X,Y)$ such that $f(K) \subset V$, and

$W(L,U)$ be the set of functions $f \varepsilon I^*(X,Y)$ such that $L \subset f(U)$.

3.7 <u>Definition</u>. The <u>interior topology</u> for $I^*(X,Y)$ is that obtained by prescribing as subbasic open sets, the totality of sets $M(K,V)$ and $W(L,U)$, where K and L are compact subsets of X and Y respectively, and U and V are open subsets of X and Y.

The aggregate of sets $M(K,V)$ alone taken as a subbasis yields the compact-open topology. $I^*(X,Y)$ under the compact-open topology is a T_i-space whenever Y is a T_i-space $(i = 0,1,2)$. Since the interior topology is finer than the compact-open topology, $I^*(X,Y)$ under the interior topology is also a T_i-space whenever Y is a T_i-space. A stronger separation property is the following.

3.8 <u>Theorem</u>. If X is a locally compact Hausdorff space and Y is a

Hausdorff space, then $I^*(X,Y)$ is a completely regular Hausdorff space under the interior topology.

Proof. It is known that a topological space (Z,T) is completely regular if and only if there exists a uniform structure $\mathfrak{F}$ for Z such that the topology T' induced in Z by $\mathfrak{F}$ is equivalent to T. To prove the theorem, it is then sufficient to show the existence of such a structure for $I^*(X,Y)$ with the required property. (Bourbaki [4,5])

Let K be a compact subset and U an open subset of X such that $K \subset U$. Define $R(K,U)$ by

$$R(K,U) = \{(f,g) \mid f(K) \cup g(K) \subset f(U) \cap g(U)\}$$

and let $\mathfrak{F}$ be the set of all finite intersections of sets $R(K,U)$, i.e. $\mathscr{R} \in \mathfrak{F}$ if and only if $\mathscr{R} = R(K_1,U_1) \cap \ldots \cap R(K_n,U_n)$ where $K_i \subset U_i$.

1) Each $\mathscr{R}$ in $\mathfrak{F}$ contains the diagonal D of $I^*(X,Y) \times I^*(X,Y)$ and $D = \bigcap_{\mathfrak{F}} \mathscr{R}$

2) If $\mathscr{R}, \mathscr{R}' \in \mathfrak{F}$, then $\mathscr{R} \cap \mathscr{R}' \in \mathfrak{F}$

3) $\mathscr{R}^{-1} = \mathscr{R}$

4) If $\mathscr{R} \in \mathfrak{F}$, then there exists a $\mathscr{T} \in \mathfrak{F}$ such that $\mathscr{T}^2 \subset \mathscr{R}$

1), 2), and 3) are direct consequences of the definition of $\mathfrak{F}$. To prove 4) it is sufficient to consider the case where $\mathscr{R} = R(K,U)$. Choose an open set V such that $K \subset V \subset \overline{V} \subset U$ and $\overline{V}$ is compact. Let $\mathscr{T} = R(K,V) \cap R(\overline{V},U)$. Suppose $(f,h) \in \mathscr{T}$ and $(g,h) \in \mathscr{T}$. Then

$$f(K) \cup h(K) \subset f(V) \cap h(V) \quad \text{and} \quad f(\overline{V}) \cup h(\overline{V}) \subset f(U) \cap h(U), \quad \text{and}$$

$$g(K) \cup h(K) \subset g(V) \cap h(V) \quad \text{and} \quad g(\overline{V}) \cup h(\overline{V}) \subset g(U) \cap h(U).$$

From these relations, it follows that $f(K) \cup g(K) \subset f(U) \cap g(U)$, i.e. (f,g) is in $R(K,U) = \mathscr{R}$ and 4) is proved. $\mathfrak{F}$ is thus a uniform structure for $I^*(X,Y)$

Let us now compare the topology induced in $I^*(X,Y)$ by $\mathfrak{F}$ and the interior topology for $I^*(X,Y)$. Suppose f is in the subbasic open set $M(K,V)\cap W(L,U)$ of the interior topology. Then $f(K)\subset V$ and $L\subset f(U)$. Choose an open set $U_1\subset X$ such that $K\subset U_1$ and $f(U_1)\subset V$, and choose a compact set $K_1\subset U$ such that $L\subset f(K_1)$. Let $\mathcal{R} = R(K,U_1)\cap R(K_1,U)$. Suppose $(f,g)\varepsilon\mathcal{R}$. Then

$$f(K)\cup g(K)\subset f(U_1)\cap g(U_1) \quad\text{and}\quad f(K_1)\cup g(K_1)\subset f(U)\cap g(U)$$

and $g\varepsilon M(K,V)\cap W(L,U)$ so the topology induced by $\mathcal{F}$ is finer than the interior topology.

Now suppose $\mathcal{R}\varepsilon\mathcal{F}$. Let $\mathcal{R} = R(K,U)$ and consider the neighborhood $\mathcal{R}(f)$ of f. $f(K)$ is compact and $f(U)$ is open since f is strongly open. Therefore $N = M(K,f(U))\cap W(f(K),U)$ is a subbasic open set of the interior topology. $f\varepsilon N$. Suppose $g\varepsilon N$. Then $g(K)\subset f(U)$ and $f(K)\subset g(U)$. Moreover since $K\subset U$, $f(K)\cup g(K)\subset f(U)\cap g(U)$, i.e. $N\subset\mathcal{R}(f)$. Thus the interior topology is finer than that induced by $\mathcal{F}$. Together with the previous conclusion, we have that the interior topology is equivalent to the topology induced in $I^*(X,Y)$ by $\mathcal{F}$. q.e.d.

3.9 <u>Corollary</u>. If X is a perfectly separable locally compact Hausdorff space and Y is a Hausdorff space, $I^*(X,Y)$ under the interior topology is metrizable. If in addition, Y is perfectly separable and locally compact, then $I^*(X,Y)$ is perfectly separable.

Proof. The first assertion is a consequence of the fact that whenever a countable subset $\mathcal{F}_0\subset\mathcal{F}$ is a base for a uniform structure $\mathcal{F}$, the topology induced by $\mathcal{F}$ is metrizable (Bourbaki [5, p. 23]). Now if X is perfectly separable and locally compact, we can construct $\mathcal{F}$ so that $\mathcal{F}$ itself is countable. To do this, let $\mathcal{U}$ be a countable base for the topology for X such that if $U\varepsilon\mathcal{U}$ then $\overline{U}$ is compact, and let $\mathcal{U}_f$ be the base consisting of finite unions of elements in $\mathcal{U}$. Let $\mathcal{R}\varepsilon\mathcal{F}$ if and only if

$\mathcal{R} = R(\overline{U}_1, U_1') \cap \dots \cap R(\overline{U}_n, U_n')$ where $U_i, U_i' \varepsilon \mathcal{U}_f$ and $\overline{U}_i \subset U_i'$. Then $\mathfrak{F}$ is countable and it is obvious that $\mathfrak{F}$ satisfies the conditions 1), 2), 3) and 4) listed in the proof of Theorem 3.8 and that the topology induced by $\mathfrak{F}$ is equivalent to the interior topology.

The second assertion can be shown by considering the interior topology itself. Let $\mathcal{V}$ be a countable base for the topology for Y such that if $V \varepsilon \mathcal{V}$ then $\overline{V}$ is compact and let $\mathcal{V}_f$ be defined as $\mathcal{U}_f$ above. Then given a function $g \varepsilon M(K,V) \cap W(L,U)$, we can choose $U_1, U_2 \varepsilon \mathcal{U}_f$ and $V_1, V_2 \varepsilon \mathcal{V}_f$ such that $g \varepsilon M(\overline{U}_1, V_1) \cap W(\overline{V}_2, U_2) \subset M(K,V) \cap W(L,U)$. The subbase obtained by using elements in $\mathcal{U}_f$ and $\mathcal{V}_f$ or their closures is countable. q.e.d.

Hark back now to the problem of finding a topology for some subset of $I(X,Y)$ under which convergence in the topological sense is consistent with interior convergence.

3.10 Theorem. Let X be a locally compact Hausdorff space and Y a Hausdorff space.

(a) If $f_0 = \lim\limits_{\Delta} f_\delta$ in $I^*(X,Y)$ under the interior topology, then $f_\delta \to f_0$ interiorly.

(b) In the subset $I^{**} \subset I^*(X,Y)$ of functions which map X onto Y, $f_0 = \lim\limits_{\Delta} f_\delta$ under the interior topology if and only if $f_\delta \to f_0$ interiorly.

Proof. Both (a) and (b) rely partially on Theorem 1.10 (b): viz. for mappings on a locally compact Hausdorff space, $f_0 = \lim\limits_{\Delta} f_\delta$ in the compact-open topology if and only if $f_\delta \to f_0$ continuously. Together with Theorem 3.5, this gives us the conclusion of part(a) by using the fact that $f_0(X)$ is a locally compact open subspace of Y.

(b) The converse implication of this part, follows by noticing that the point $y_0 \varepsilon f_0(U)$ which occurs in the statement of Theorem 3.5 may be replaced by a compact set $L \subset f_0(U)$ and also that here we are dealing with functions which map X onto Y. q.e.d.

The converse of Theorem 3.10 (a) may be shown whenever Y is locally connected.

3.11 Lemma. Let $f\colon X \to Y$ be a continuous map of a space X into a Hausdorff space Y such that $f(X)$ is open in Y. If U is a subset of X with compact closure and C a connected subset of Y such that $\emptyset \neq C \cap f(X) \subset f(U)$, then $C \subset f(U)$.

Proof. Since $C \cap f(X) \subset f(U)$ and $\overline{U}$ is compact, we have $\overline{C \cap f(X)} \subset \overline{f(U)} = f(\overline{U}) \subset f(X)$. Hence $C \cap \overline{C \cap f(X)} \subset C \cap f(X)$. But $C \cap f(X) = C \cap C \cap f(X) \subset C \cap \overline{C \cap f(X)}$ and so $C \cap \overline{C \cap f(X)} = C \cap f(X)$, i.e. $C \cap f(X)$ is closed in C. By hypothesis $f(X)$ is open in Y. Hence $C \cap f(X)$ is also open in C. C is connected. Therefore $C = C \cap f(X) \subset f(U)$. q.e.d.

3.12 Theorem. Let X be a locally compact Hausdorff space and Y a locally connected Hausdorff space. Then $f_0 = \lim_{\Delta} f_\delta$ in $I^*(X,Y)$ under the interior topology if and only if $f_\delta \to f_0$ interiorly.

Proof. Suppose $f_\delta \to f_0$ interiorly. Let U be an open subset of X and L a compact subset of Y such that $L \subset f_0(U)$, i.e. suppose $f_0 \varepsilon W(L,U)$. For each point $y_i \varepsilon L$ choose a point $x_i \varepsilon U \cap f_0^{-1}(y_i)$ and a neighborhood U_i of x_i such that $\overline{U}_i \subset U$ and $\overline{U}_i$ is compact. By Theorem 3.5, there is a neighborhood V_i of y_i and a δ_i such that $V_i \cap f_\delta(X) \subset f_\delta(U_i)$ whenever $\delta > \delta_i$. We may suppose that V_i is connected and $V_i \cap f_\delta(X) \neq \emptyset$. Then by Lemma 3.11, $V_i \subset f_\delta(U_i)$ whenever $\delta > \delta_i$. Since L is compact there is a $\delta_0 \varepsilon \Delta$ such that for $\delta > \delta_o$, $L \subset f_\delta(U)$, i.e. $f_\delta \in W(L,U)$. This proves that $f_o = \lim_\Delta f_\delta$ in the interior topology. q.e.d.

Remark. In the example following Theorem 3.4, $f_o, f_n, g_{nm} \in I^*(X,Y)$ where $X = Y$ is not locally connected at the point 0. There it has been shown that for the set $G = \{g_{nm} \mid m,n = 1,2,3,\ldots\}$, $\overline{\overline{G}} \neq \overline{G}$, $\overline{G}$ being defined in terms of interior convergence as before. This shows that the condition of local connectedness of the space Y occurring in Theorem 3.12 is not entirely superfluous. By modifying the example, it is possible to show that, even if Y is locally connected, in general there is no topology for $I(X,Y)$ such that convergence in the topological sense is consistent with interior convergence, i.e. $I^*(X,Y)$ in Theorem 3.12 cannot be replaced by the larger set $I(X,Y)$ and a new topology for $I(X,Y)$. For take X to be the same space of the previous example and now let $Y = \{y \mid 0 \leq y,\ y \text{ is real}\}$. Then $f_o, f_n, g_{nm} \in I(X,Y)$, and if again we let $G = \{g_{nm} \mid m,n = 1,2,3,\ldots\}$, then $\overline{\overline{G}} \neq \overline{G}$ as before.

By Theorem 3.6, if $f_\delta \to f_o$ interiorly, then the limit map f_o is interior. By modifying the proof of Theorem 3.12 slightly, it can be shown that if each f_δ is strongly interior, then f_o is also strongly interior.

3.13 Theorem. Let X be a locally compact Hausdorff space and Y a locally connected Hausdorff space. If $f_\delta \to f_o$ interiorly, where each $f_\delta \in I^*(X,Y)$, then $f_o \in I^*(X,Y)$.

Proof. Let U be an open subset of X and suppose $y \in f_o(U)$. As in the proof of Theorem 3.12, there is an open set U' such that $\overline{U'} \subset U$ and $\overline{U'}$ is compact, an open set V of Y and a $\delta_o \in \Delta$ such that for $\delta > \delta_o$, $y \in V \subset f_\delta(U')$. $f_o(\overline{U'}) = \lim_\Delta f_\delta(\overline{U'})$ since $f_\delta \to f_o$ continuously. Hence $y \in V \subset f_o(\overline{U'}) \subset f_o(U)$. q.e.d.

IV. INTERIOR MAPS INTO LOCALLY CONNECTED SPACES

Let X be a locally compact metric space and Y a locally connected metric space. From Theorem 3.12, it follows that in $I^*(X,Y)$, $f_n \longrightarrow f_o$ interiorly if and only if $f_n \longrightarrow f_o$ continuously and the sequence $\{f_n \mid n = 1,2,3,\ldots\}$ is uniformly approximately interior on compact sets. (See Whyburn [15, p. 20] and remark following Theorem 3.5, this paper.) This section follows rather closely G. T. Whyburn's investigations of such sequences and related subjects. What novelty there is here consists mainly in relaxing conditions such as perfect separability and metrizability of the spaces considered except in Theorem 4.12, where certain results of analytic function theory are adduced. All functions considered in this section are assumed to be continuous.

By Theorem 3.12, if $f_\delta \longrightarrow f_o$ continuously, as far as strongly interior mappings f_δ on a locally compact Hausdorff space X into a locally connected Hausdorff space Y are concerned, the following condition is necessary and sufficient for interior convergence:

$$\text{If } y_o = \lim_{\Sigma} y_\sigma, \text{ then } f_o^{-1}(y_o) = \operatorname*{Lim}_{\Delta\times\Sigma} f_\delta^{-1}(y_\sigma).$$

This condition is a strengthened version of the conclusion of a theorem of Hurwitz concerning analytic functions of a complex variable:

> Let f_n be a sequence of functions, each analytic in a region D bounded by a simple closed contour, and let $f_n \longrightarrow f_o$ uniformly in D. Suppose that f_o is not identically zero. Let z_o be an interior point of D. Then z_o is a zero of f_o, if and only if it is a limit-point of the set of zeros of the functions f_n, points which are zeoos for an infinity of values of n being counted as limit-points.
>
> Titchmarsh [14, p. 119]

Taking the region D to be open, the last sentence of the theorem quoted above may be rendered symbolically in the notation we have adopted as

$$D \cap f_0^{-1}(0) = D \cap \operatorname*{Lim}_{n\to\infty} f_n^{-1}(0).$$

Among other things, in this section we shall seek a generalization of Hurwitz's theorem.

4.1 Definition. The boundary of a subset A of a topological space X is the set $\text{Bdry}(A) = \overline{A} \cap \overline{(X - A)}$.

4.2 Definition. The interior of a subset A of a topological space X is the set $\text{Int}(A) = X - \overline{(X - A)}$.

$\overline{A} = \text{Bdry}(A) \cup \text{Int}(A)$. $\text{Bdry}(A) \cap \text{Int}(A) = \emptyset$.

4.3 Definition. A subset A of a space is conditionally compact provided $\overline{A}$ is compact.

The following lemma, stated here without proof, is a variant of a theorem proved in R. L. Wilder's Topology of Manifolds [16, p. 100, Theorem 1.2]

4.4 Lemma. If M is a compact component of a closed subset F of a locally compact Hausdorff space X and U is an open set of X containing M, then there exists a conditionally compact open set V of X such that $M \subset V \subset \overline{V} \subset U$ and $F \cap \text{Bdry}(V) = \emptyset$.

4.5 Definition. A mapping $f\colon X \to Y$ is said to be quasi-interior provided for any $y \in Y$ and open set U of X containing a compact component of $f^{-1}(y)$, $y \in \text{Int}(f(U))$. (Whyburn [15, p. 9])

Every strongly interior mapping is quasi-interior.

4.6 Theorem. Let $f: X \to Y$ be a mapping on a locally compact Hausdorff space X into a Hausdorff space Y. Then each of the following conditions is equivalent to quasi-interiority of f:

(a) For each conditionally compact set $A \subset X$, $\mathrm{Bdry}(f(A)) \subset f(\mathrm{Bdry}(A))$.

(b) For each compact set $K \subset X$, $\mathrm{Bdry}(f(K)) \subset f(\mathrm{Bdry}(K))$.

Proof. 1) Quasi-interiority implies (a). Suppose there is a conditionally compact subset A of X such that $\mathrm{Bdry}(f(A)) \not\subset f(\mathrm{Bdry}(A))$. By the continuity of f, the compactness of $\overline{A}$, and the assumption that Y is a Hausdorff space, we have $\overline{f(A)} = f(\overline{A})$. $\mathrm{Bdry}(f(A)) \subset \overline{f(A)} = f(\overline{A})$ and $f(\overline{A}) = f(\mathrm{Bdry}(A) \cup \mathrm{Int}(A)) = f(\mathrm{Bdry}(A)) \cup f(\mathrm{Int}(A))$. Hence there is a point $x \in \mathrm{Int}(A)$ such that $f(x) \in \mathrm{Bdry}(f(A)) - f(\mathrm{Bdry}(A))$. Let $y = f(x)$. Then $f^{-1}(y) \cap \mathrm{Bdry}(A) = \emptyset$. A is conditionally compact and hence $\mathrm{Int}(A)$ also. There is a compact component of $f^{-1}(y)$ contained in $\mathrm{Int}(A)$. But $y \notin \mathrm{Int}(f(\mathrm{Int}(A)))$ since $\mathrm{Int}(f(\mathrm{Int}(A))) \subset \mathrm{Int}(f(A))$, $\mathrm{Bdry}(f(A)) \cap \mathrm{Int}(f(A)) = \emptyset$, and $y \in \mathrm{Bdry}(f(A))$. Hence f is not quasi-interior.

2) (a) implies (b) since any compact subset of a Hausdorff space is conditionally compact.

3) (b) implies f is quasi-interior. Suppose L is a compact component of $f^{-1}(y)$ and suppose U is an open set containing L. Let V be a conditionally compact open set such that $L \subset V \subset \overline{V} \subset U$ and $f^{-1}(y) \cap \mathrm{Bdry}(V) = \emptyset$. Then $f^{-1}(y) \cap \mathrm{Bdry}(\overline{V}) = \emptyset$ since $\mathrm{Bdry}(\overline{V}) \subset \mathrm{Bdry}(V)$. By hypothesis, $\mathrm{Bdry}(f(\overline{V}))$ is contained in $f(\mathrm{Bdry}(\overline{V}))$. But $y \notin f(\mathrm{Bdry}(\overline{V}))$ and therefore $y \notin \mathrm{Bdry}(f(\overline{V}))$. $y \in \overline{f(V)} = \mathrm{Bdry}(\overline{f(V)}) \cup \mathrm{Int}(\overline{f(V)}) = \mathrm{Bdry}(f(\overline{V})) \cup \mathrm{Int}(f(\overline{V}))$. Hence $y \in \mathrm{Int}(f(\overline{V})) \subset \mathrm{Int}(f(U))$. q.e.d.

The next theorem is a generalization of G. T. Whyburn's topological analog of the Weierstrass double series theorem (Whyburn [15, p. 10]). The compact-open topology for $C(X,Y)$, the set of all continuous functions on X into Y, is used here instead of the topology (used by Whyburn) of uniform convergence on compact sets, to which the compact-open topology is equivalent whenever Y is a uniform space (Arens [2, Theorem 9]).

4.7 <u>Theorem</u>. The set of quasi-interior mappings on a locally compact Hausdorff space X into a locally connected Hausdorff space Y is closed in $C(X,Y)$ under the compact-open topology.

Proof. Suppose $f\colon X \to Y$ is not quasi-interior. By condition (b) of Theorem 4.6, there is a compact set $K \subset X$ such that $\mathrm{Bdry}(f(K)) \not\subset f(\mathrm{Bdry}(K))$. Then, as shown in part 1) of the proof of Theorem 4.6, there is a point $x_1 \varepsilon \mathrm{Int}(K)$ such that $y_1 = f(x_1) \varepsilon \mathrm{Bdry}(f(K)) - f(\mathrm{Bdry}(K))$.

Case I. $\mathrm{Bdry}(K) \neq \emptyset$. Since $f(\mathrm{Bdry}(K))$ is closed and compact, and Y is locally connected, there is a pair of open subsets W_1 and W_2 of Y, such that $y_1 = f(x_1) \varepsilon W_1$, $f(\mathrm{Bdry}(K)) \subset W_2$, $W_1 \cap W_2 = \emptyset$, and such that W_1 is connected. Now since $y_1 \varepsilon \mathrm{Bdry}(f(K))$, $W_1 \not\subset f(K)$. Hence there is an open set W_3 such that $f(K) \subset W_3$ but such that $W_1 \not\subset W_3$. Consider the set

$$N = M(x_1, W_1) \cap M(\mathrm{Bdry}(K), W_2) \cap M(K, W_3)$$

of continuous functions. N is an open set in $C(X,Y)$ under the compact-open topology and $f \varepsilon N$.

Suppose $g \varepsilon N$. Then $g(K) \subset W_3$ and since $x_1 \varepsilon K$, $g(K) \cap W_1 \neq \emptyset$. But $W_1 \not\subset W_3$. Therefore $W_1 \cap (Y - g(K)) \neq \emptyset$. W_1 is connected. Hence $W_1 \cap \mathrm{Bdry}(g(K)) \neq \emptyset$. $g(\mathrm{Bdry}(K)) \subset W_2$ and $W_1 \cap W_2 = \emptyset$. Therefore $\mathrm{Bdry}(g(K)) \not\subset g(\mathrm{Bdry}(K))$ and by Theorem 4.6, g is not quasi-interior.

Case II. $\mathrm{Bdry}(K) = \emptyset$. Let $N' = M(x_1, W_1) \cap M(K, W_3)$, where $W_1 \not\subset W_3$, $f(x_1) \varepsilon \mathrm{Bdry}(f(K))$, and W_1 is connected as in Case I. Then $f \varepsilon N'$ and if $g \varepsilon N'$ then $\mathrm{Bdry}(g(K)) \neq \emptyset$. Again $\mathrm{Bdry}(g(K)) \not\subset g(\mathrm{Bdry}(K))$ since the latter is empty, and g is not quasi-interior. q.e.d.

A simple example serves to show that the condition of local connectedness of the space Y appearing in the above theorem is not superfluous. Let $X = \{x \mid x = 0 \text{ or } x = 1/n,\ n = 1,2,3,\ldots\}$ and consider the sequence of constant functions $f_n\colon X \to X$ defined by $f_n(x) = 1/n$. Each f_n is quasi-interior and the sequence converges in the compact-open topology to the constant function $f_0(x) = 0$. But f_0 is not quasi-interior.

4.8 Definition. A mapping $f\colon X \to Y$ is said to be light provided that for each $y \varepsilon Y$, $f^{-1}(y)$ is totally disconnected.

A light quasi-interior mapping is strongly interior.

4.9 Lemma. Let $f\colon X \to Y$ be a light strongly interior mapping on a locally compact Hausdorff space X into a locally connected Hausdorff space Y. Let U be an open subset of X and L a compact subset of Y such that $L \subset f(U)$. Then there exist an open set W of Y containing L and a neighborhood N of f of the compact-open topology, such that if $g \varepsilon N$ and g is quasi-interior, then $L \subset W \subset g(U)$.

Proof. Let us establish the theorem for the case where L consists of a single point. The extension to the general case where L is a non-degenerate compact set is then obvious.

Let $L = \{y_0\} \subset f(U)$. If y_0 is an isolated point of Y, then $\{y_0\}$ is an open set in Y and we can easily specify a neighborhood N of f which satisfies the conclusion of the lemma. Suppose y_0 is not isolated in Y. Let R be a connected open set such that $y_0 \varepsilon R$ and such that if A is any non-empty subset of R, then $\mathrm{Bdry}(A) \neq \emptyset$. This is always possible since Y is locally connected. Choose an $x_0 \varepsilon f^{-1}(y_0) \cap U$. Since f is light and continuous, there is a neighborhood V_0 of x_0, such that $V_0 \subset U$, $\overline{V}_0$ is compact, $f(V_0) \subset R$, and $f^{-1}(y_0) \cap \mathrm{Bdry}(V_0) = \emptyset$. By Theorem 4.6 (a), $\mathrm{Bdry}(f(V_0)) \subset f(\mathrm{Bdry}(V_0))$ and by our choice of R, $\mathrm{Bdry}(f(V_0)) \neq \emptyset$, and hence $\mathrm{Bdry}(V_0) \neq \emptyset$. Now $f(\mathrm{Bdry}(V_0))$ is compact and $y_0 \notin f(\mathrm{Bdry}(V_0))$. Choose a connected neighborhood W_0 of y_0 such that $\overline{W}_0 \cap f(\mathrm{Bdry}(V_0)) = \emptyset$.

Consider the open set

$$N = M(x_0, W_0) \cap M(\mathrm{Bdry}(V_0), Y - \overline{W}_0)$$

of the compact-open topology. $f \varepsilon N$. Suppose $g \varepsilon N$ and g is quasi-interior. Then $\mathrm{Bdry}(g(V_0)) \subset g(\mathrm{Bdry}(V_0)) \subset Y - \overline{W}_0$, and hence $W_0 \cap \mathrm{Bdry}(g(V_0)) = \emptyset$. But $x_0 \varepsilon V_0$, therefore $W_0 \cap g(V_0) \neq \emptyset$. W_0 is connected. Hence $W_0 \subset g(V_0) \subset g(U)$. q.e.d.

4.10 <u>Theorem</u>. Let f_δ, $\delta \varepsilon \Delta$ be a directed set of quasi-interior mappings on a locally compact Hausdorff space X into a locally connected Hausdorff space Y. If $f_\delta \to f_0$ continuously, where f_0 is a light mapping, then f_0 is strongly interior and $f_\delta \to f_0$ interiorly.

Proof. If $f_\delta \to f_0$ continuously, then $f_0 = \lim_{\Delta} f_\delta$ in $C(X,Y)$ under the compact-open topology. By Theorem 4.7, f_0 is quasi-interior. But a light quasi-interior mapping is strongly interior. By Theorem 3.5 and Lemma 4.9, $f_\delta \to f_0$ interiorly. q.e.d.

4.11 <u>Theorem</u>. If $L(X,Y)$ is the set of light strongly interior mappings on a locally compact Hausdorff space X into a locally connected Hausdorff space Y, then the compact-open topology and the interior topology for $L(X,Y)$ are equivalent.

The above theorem is a direct consequence of the definition of the interior topology and Lemma 4.9.

Local connectedness in Theorems 4.10 and 4.11 is not redundent. Arens [3, p. 601] has given an example of a locally compact Hausdorff space which is not locally connected at a single point and in which there is a sequence of homeomorphisms h_n such that $h_0 = \lim_{n\to\infty} h_n$ in the compact-open topology but $h_0^{-1} \neq \lim_{n\to\infty} h_n^{-1}$. We cannot expect Theorems 4.10 or 4.11 to be valid for such spaces since in the set $H(X)$ of homeomorphisms onto, the interior topology and Arens' g-topology are the same, and $h_0 = \lim_{n\to\infty} h_n$ in the g-topology if and only if $h_0 = \lim_{n\to\infty} h_n$ and $h_0^{-1} = \lim_{n\to\infty} h_n^{-1}$ in the compact-open topology.

Examples. In Theorem 4.10, the strong interiority of f_0 is obtained immediately by means of Theorem 4.7 and the assumption that f_0 is light. As a possible generalization of Theorem 4.10, it may be thought that the lightness of f_0 may be replaced by strong interiority of f_0 in the hypothesis of the theorem. This is not true in general. To settle other questions that may naturally arise in connection with Theorem 4.10, we may even require that every map f_δ in the directed set to have finite point inverses, to be strongly interior, and onto, the limit map f_0 to be strongly interior and onto, the domain and range space to be the same locally connected and locally compact Hausdorff space, and $f_\delta \to f_0$ continuously. The following example shows that even with these seemingly stringent conditions, without the lightness of f_0, the directed set of functions f_δ may not converge interiorly to f_0 and indeed the inverse functions f_δ^{-1} may not converge pointwise to the function f_0^{-1}.

Let $Q_0 = \{(x,y) \mid 0 \leq x,\ 0 \leq y\}$ be the closed first quadrant of the plane with the usual topology. Define $f_0\colon Q_0 \to Q_0$ by

$$f_0(x,y) = \begin{cases} (x|\cos(2\pi y/x)|,\ x|\sin(2\pi y/x)|) & \text{for } x \neq 0 \\ (0,0) & \text{for } x = 0 \end{cases}$$

f_0 is a strongly interior map of $\mathcal{Q}_0$ onto $\mathcal{Q}_0$. For each $n = 1,2,3,\ldots$ define $f_n\colon \mathcal{Q}_0 \longrightarrow \mathcal{Q}_0$ by

$$f_n(x,y) = \begin{cases} \left(\left(x+\frac{y}{n}\right)\left|\cos\left(\frac{2\pi y}{x+\frac{y}{n}}\right)\right|,\ \left(x+\frac{y}{n}\right)\left|\sin\left(\frac{2\pi y}{x+\frac{y}{n}}\right)\right|\right) & \text{for } (x,y)\neq(0,0) \\ (0,0) & \text{for } (x,y) = (0,0) \end{cases}$$

Then $f_n \rightarrow f_0$ continuously. That each f_n is a strongly interior mapping of $\mathcal{Q}_0$ onto itself with finite point inverses may be demonstrated in the following manner: Let $\mathcal{Q}_n \subset \mathcal{Q}_0$ be the closed sector subtended by the x-axis and the straight line passing through $(0,0)$ and having slope n. Then f_n may be defined as the composition of two maps, the homeomorphism $h_n\colon \mathcal{Q}_0 \rightarrow \mathcal{Q}_n$ defined by $h_n(x,y) = (x+y/n, y)$ followed by the partial map $f_0|\mathcal{Q}_n$. Examination of the original map f_0 reveals that each of the partial maps $f_0|\mathcal{Q}_n$, maps $\mathcal{Q}_n$ onto $\mathcal{Q}_0$ continuously, and moreover each $f_0|\mathcal{Q}_n$ is a strongly interior map having finite point inverses and hence each f_n also shares these properties. But $f_0^{-1}(0,0) = \{(0,y) \mid 0 \leq y\}$ and $f_n^{-1}(0,0) = (0,0)$. Hence $f^{-1}(0,0) \neq \operatorname{Lim}_{n\to\infty} f^{-1}(0,0)$.

In the example just given, f_0 fails to be light because $f_0^{-1}(0,0)$ consists of a single non-compact component. Suppose now we say that a map $f\colon X \rightarrow Y$ has property A if for every $y \in Y$, each component of $f^{-1}(y)$ is compact, and attempt to generalize Theorem 4.10 by replacing the light map f_0 by a strongly interior one having property A, leaving the remaining hypothesis unchanged. Let $I = \{x \mid 0 \leq x \leq 1\}$. For each $n = 1,2,3,\ldots$ define $f_n\colon I \times I \rightarrow I$ by

$$f_n(x,y) = x(1 - y/(n+1))$$

Then f_n converges continuously to the projection $f_0\colon I \times I \rightarrow I$, $f_0(x,y) = x$.

f_o and f_n for each n are strongly interior maps having property A. But $f_n^{-1}(1) = (1,0)$ for each n, while $f_o^{-1}(1) = \{(1,y) \mid 0 \leq y \leq 1\}$. Again $f_o^{-1}(1) \neq \operatorname{Lim}_{n\to\infty} f_n^{-1}(1)$ and f_n does not converge interiorly to f_o. The lightness of f_o in Theorem 4.10 seems to be a crucial property.

Using some of our results on interior convergence and some theorems on analytic functions, we are now in a position to strengthen the conclusion of Hurwitz's theorem and prove a converse. The theorems on analytic functions alluded to are (1) Weierstrass' double series theorem: if $f_n \to f_o$ uniformly in an open region D and f_n is analytic for each n, then f_o is itself analytic, and (2) Stoilow's theorem: a non-constant analytic function on an open region D is both strongly interior and light on D (Stoilow [13]).

4.12 <u>Generalized Hurwitz's Theorem</u>. Let $\{f_n\}$ be a sequence of functions, each analytic in a bounded open region D and let $f_n \to f_o$ uniformly in D. Then f_o is non-constant if and only if whenever $w_o = \lim_{n\to\infty} w_n$, $f_o^{-1}(w_o) = \operatorname{Lim}_{n\to\infty} f_n^{-1}(w_n)$ in D.

Proof. Necessity. Let W be the entire complex plane. By the Weierstrass double series theorem and Stoilow's result, $f_o: D \to W$ is strongly open and light. Since f_o is non-constant, we may suppose that each f_n is non-constant, and therefore strongly open and light. Now if $f_n \to f_o$ uniformly, then $f_o = \lim_{n\to\infty} f_n$ in $I^*(D,W)$ in the compact-open topology. Then by Theorem 4.11, $f_o = \lim_{n\to\infty} f_n$ in the interior topology and hence if $w_o = \lim_{n\to\infty} w_n$, then $f_o^{-1}(w_o) = \operatorname{Lim}_{n\to\infty} f_n^{-1}(w_n)$.

Sufficiency. $f_o^{-1}(w_o) = \operatorname{Lim}_{n\to\infty} f_n^{-1}(w_n)$ in D whenever $w_o = \lim_{n\to\infty} w_n$, means that the inverse functions f_n^{-1} on W into 2^D converge continuously to f_o^{-1}. D is a locally compact Hausdorff space. Therefore 2^D is regular and hence

f_0^{-1} is continuous (Theorem 1.8), i.e. f_0 is strongly open and hence is a non-constant analytic function. q.e.d.

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VITA

Shigeru Ishii was born on December 9, 1921 in Honolulu, Territory of Hawaii. After he was graduated from the Wallace Rider Farrington High School of Honolulu in 1940, he entered the University of Hawaii. His undergraduate work there was interrupted on December 7, 1941 when the University of Hawaii was closed after the Japanese attack on Pearl Harbor. In 1943 he enlisted in the Army of the United States and served in the American, European, and Asiatic-Pacific Theaters. Discharged from the Army in 1945, he reentered the University of Hawaii in 1946. In 1947 he transferred to the University of Illinois where he received the degrees of Bachelor of Arts in February, 1949 and Master of Arts in August, 1949.